

ON THE LACUNARITY OF SOME ETA-PRODUCTS

YUDONG WANG, CHUNLEI LIU, HAobo DAI

ABSTRACT. The lacunarity is an interesting property of a formal series. We say a series is lacunary if "almost all" of its coefficients are zero. In this article we considered about the lacunarity of some eta-products like $\eta(z)^2\eta(bz)^2$, and proved that they are lacunary if and only if $b \in \{1, 2, 3, 4, 16\}$. Then We write them as linear combinations of some CM forms.

1. INTRODUCTION

The Dedekind eta-function, defined as:

$$\eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$$

where $q = 2\pi iz$, plays an important role in modular form theories. Moreover, as the generating function of the Partition Function $p(n)$:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-1}$$

$\eta(z)$ is also very useful in partition theory and combinatorial theory. So are the eta-quotients, defined as the products and quotients of some eta functions:

$$f(z) := \prod_{\delta} \eta(\delta z)^{r_{\delta}} = q^s \prod_{n=1}^{\infty} \prod_{\delta} (1 - q^{\delta n})^{r_{\delta}}$$

where δ are positive integers, and r_{δ} , related to δ , are non-zero integers. $s := \frac{1}{24} \sum_{\delta} \delta r_{\delta}$. When all r_{δ} are positive, $f(z)$ is also called eta-products.

An interesting property of a series $\sum_{n=0}^{\infty} a(n)q^n$ is the "lacunarity". A series $\sum_{n=0}^{\infty} a(n)q^n$ is said to be *lacunary* if "almost all" of its coefficients are zero, that is:

$$\lim_{X \rightarrow \infty} \frac{\#\{0 \leq n \leq X : a(n) = 0\}}{X} = 1$$

By far, there have been many researches on the lacunarity of eta-quotients. For example in [1], Serre proved that given a positive even integer r , $\eta(z)^r$ is lacunary if and only if $r \in \{2, 4, 6, 8, 10, 14, 26\}$. In [2], Clader discussed the eta-quotients in the form $\frac{\eta(az)^b}{\eta(z)}$, and proved the only 19 cases when they are lacunary. In [3], Martin researched the lacunarity of eta-quotients which are Hecke eigenforms. In this article, we discuss the eta-products in the form $\eta(z)^2\eta(kz)^2$, and prove the following theorem:

Theorem 1 Let $f_b(z) := \eta(z)^2\eta(bz)^2$, b is a positive integer and has no square integer divisor except 2,3. $23 \nmid b$. Then $f_b(z)$ is lacunary if and only if $b \in \{1, 2, 3, 4, 16\}$.

1991 *Mathematics Subject Classification.* 11N13, 11B25.

This work is supported by Project 11071160 of the Natural Science Foundation of China.

We will prove this theorem in section 3, using the theory of modular form and CM forms. After that, we will give the method of computing the coefficients of $f_b(z)$ when it is lacunary in section 4.

2. PRELIMINARIES

The Dedekind eta-function $\eta(24z)$, as we know, is a weight $\frac{1}{2}$ modular form on $\Gamma_0(576)$, with Nebentypus character $\chi_{12}(n) := \left(\frac{12}{n}\right)$. As consequence, the eta-quotient may be modular form on some modular group. In fact, we have the following theorem from Ono[4]:

Theorem 2.1 If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an eta-quotient with $k = \frac{1}{2} \sum_{\delta|N} r_\delta$, with the additional properties that

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}$$

and

$$\sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$$

then $f(z)$ satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for every $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Here the character χ is defined by $\chi(d) := \left(\frac{(-1)^k \cdot s}{d}\right)$, where $s := \prod_{\delta|N} \delta^{r_\delta}$. Moreover, if $f(z)$ is holomorphic (resp. vanishes) at all of the cusps of $\Gamma_0(N)$, then $f(z) \in M_k(\Gamma_0(N), \chi)$ (resp. $S_k(\Gamma_0(N), \chi)$).

After theorem 2.1, we can easily get the following lemma about $f_b(z)$:

Lemma 2.2 Let $f_b(12z) := \eta(12z)^2 \eta(12bz)^2$ as above. Then $f_b(12z) \in S_2(\Gamma_0(144b))$.

We still need the theory of CM forms. The basic idea of CM form can be found in Ribet[5]. Let K be an imaginary quadratic field of discriminant D , and ε_K be the quadratic character associated to K . Next, let c be a Hecke character of K with exponent $k-1$ and conductor f_c . That means, view c as a homomorphism:

$$(fractional \text{ ideals of } K \text{ prime to } f_c) \rightarrow C^*$$

we have

$$c((\alpha)) = \alpha^{k-1}$$

for all $\alpha \in F^*$ such that $\alpha \equiv 1 \pmod{f_c}$.

Next, associated to c , define a Dirichlet character $\omega_c \bmod N(f_c)$ as:

$$\omega_c(n) = c((n))/n^{k-1}$$

for all $n \in \mathbb{Z}$ coprime to f_c . $N(f_c)$ denotes the norm of f_c .

For any $\delta \in \mathbb{Z}^+$, define series $\varphi_{K,c,\delta}$ as:

$$\varphi_{K,c,\delta} = \sum_a c(a) q^{\delta \cdot N(a)}$$

where the sum runs through all integral ideals of K coprime to f_c . Then, by theorem 3.4 in [5], $\varphi_{K,c,\delta}$ is a cusp form of weight k and character $\omega_c \cdot \varepsilon_K$ on $\Gamma_0(\delta \cdot |D| \cdot N(f_c))$,

moreover, $\varphi_{K,c,\delta}$ is an eigenform for all Hecke operators T_p in which $p \nmid \delta \cdot |D| \cdot N(f_c)$. So, in order to make $\varphi_{K,c,\delta}$ to be an element in $S_k(\Gamma_0(N), \varepsilon)$, the following two conditions are necessary and sufficient:

$$\delta \cdot |D| \cdot N(f_c) \mid N$$

$$\omega_c \cdot \varepsilon_K = \varepsilon$$

All possible $\varphi_{K,c,\delta}$ satisfying the above two conditions generate a subspace of $S_k(\Gamma_0(N), \varepsilon)$, which denoted as $S_k^{cm}(\Gamma_0(N), \varepsilon)$. Then, by Serre[6], an element of $S_k(\Gamma_0(N), \varepsilon)$ is lacunary if and only if it is an element of $S_k^{cm}(\Gamma_0(N), \varepsilon)$.

3. PROOF OF THEOREM 1

With the above theory, we know that $f_b(12z)$ is lacunary if and only if $f_b(12z) = \sum \varphi_i \in S_2^{cm}(\Gamma_0(144b), \varepsilon)$, where φ_i are some suitable $\varphi_{K,c,\delta}$. First we have the following lemmas:

Lemma 3.1 $f_b(12z) = \sum \varphi_i$ as above. b as in Theorem 1. Then as a CM form, φ_i can only be associated to one of the following four fields: $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-6})$.

Proof Assume φ_i is generated by $\mathbb{Q}(\sqrt{D})$, k , f_c and ε_K , ω_c . So we have $|D| \cdot N(f_c) \mid 144b$, and $\omega_c \cdot \varepsilon_K = 1_N$, where 1_N is the trivial character on $\Gamma_0(144b)$. By $\omega_c \cdot \varepsilon_K = 1_N$, if there is some prime $p \neq 2, 3$ such that $p \mid |D|$, that is p is a divisor of the module of character ε_K , so p must be a divisor of the module of character ω_c , too. But we already know that ω_c is a Dirichlet character mod $N(f_c)$, so there must be $p \mid N(f_c)$, and $p^2 \mid |D| \cdot N(f_c) \mid 144b$. This is a contradiction to our assumption that b is a square-free integer. This completes the proof. \square

Lemma 3.2 φ_i is a CM form associated to $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-3})$ or $\mathbb{Q}(\sqrt{-6})$. T_{23} is the usual Hecke operator T_p with $p = 23$. Then $\varphi_i|T_{23} = 0$.

This lemma is obvious because 23 remains prime in each of those four fields, so the 23-rd coefficient of φ_i must be zero. Combined with the fact that φ_i is an eigenform of T_{23} , and the computing formula of T_p : $a_n(T_p(f)) = a_{np}(f) + \chi(p)p^{k-1}a_{n/p}(f)$, this lemma can be easily obtained. \square

Lemma 3.3 $f_b(12z) = \sum \varphi_i$ as in Theorem 1. Then if $b \geq 176$, $f_b(12z)$ cannot be lacunary.

Proof By Lemma 3.2, if $f_b(12z)$ is lacunary, then $f_b(12z)|T_{23} = 0$. We will show that if b is large enough, $f_b(12z)|T_{23}$ cannot be zero. Denote $f_b(12z) = q^{1+b} \prod_{n=1}^{\infty} (1 - q^{12n})^2 (1 - q^{12bn})^2 = \sum_{n=1}^{\infty} a(n)q^n$ and $\prod_{n=1}^{\infty} (1 - q^n)^2 = \sum_{n=1}^{\infty} b(n)q^n$. If $n < 12b$, then $a(n)$ has nothing to do with $(1 - q^{12bn})^2$, so we have $a(n) = b(\frac{n-(1+b)}{12})$. Denote $T_{23}(f) = \sum_{n=1}^{\infty} c(n)q^n$, Using the formula of T_p , we have $c(n) = a(23n) + p \cdot a(n/23)$. When $23 \nmid n$, $c(n) = a(23n) = b(\frac{23n-(1+b)}{12})$. In order to make $\frac{23n-(1+b)}{12}$ an integer, we need $n \equiv -(1+b) \pmod{12}$. If there exists some $c(n) \neq 0$, then $f_b(12z)|T_{23} \neq 0$. In summary, after three conditions: (1) $1+b \leq n < 12b$; (2) $23 \nmid n$; (3) $n \equiv -(1+b) \pmod{12}$; If there exists such n that $b(\frac{23n-(1+b)}{12}) \neq 0$, then $f_b(12z)$ cannot be lacunary.

Let n_0 be the smallest n that makes $\frac{23n-(1+b)}{12}$ an positive integer. So $\frac{23n-(1+b)}{12} \in \{1, 2, 3, \dots, 23\}$, and when $n_0 \rightarrow n_0 + 12$, $\frac{23n-(1+b)}{12} \rightarrow \frac{23n-(1+b)}{12} + 23$. Obviously, in

$n_0 + 12t$, $t = 0, 1, 2, \dots, 22$, these 23 numbers, only one can be divided by 23. We list the first 1000 coefficients of $b(n)$ in appendix 1.

From the table of $b(n)$ above, we only need $t \geq 5$, then there will be at least two numbers not equal to zero in each column, except the 21st column, which we will discuss later. So there must be at least one number that suits $23 \nmid n$. In order to suit $1 + b \leq n < 12b$, we need $b > \frac{12 \cdot 7 \cdot 23}{11}$, that is $b \geq 176$. As to the 21st column, in this condition, $\frac{23n_0 - (1+b)}{12} = 21$, that is $21n_0 - (1+b) = 21 \cdot 12$. Taking module of 23 on the both sides, we get $k \equiv 0 \pmod{23}$, which contradicts our assumption in Theorem 1. In summary, we can say that when $b \geq 176$, $f_b(12z)$ cannot be lacunary. \square

Now, in order to prove Theorem 1, we only need to check the lacunarity of each case of $b < 176$ by computer. The method is to check whether $f_b(12z)|_{T_{23}}$ (or T_{47} , when $23|b$) is zero. At last we find that $f_b(12z)$ cannot be lacunary except $b = 1, 2, 3, 4$ or 16 . We will prove those five cases in Proposition 4.1. Theorem 1 is proved.

4. COMPUTATION WHEN $b = 1, 2, 3, 4, 16$

When $b=1,2,3,4,16$, $f_b(z)$ is lacunary. So it can be expressed as a linear combination of CM forms. We list these specific expressions in Proposition 4.1:

Proposition 4.1

(1) $b = 1$; $f_b(6z) = \eta(6z)^4 = \varphi_{K,c}(z) \in S_2(\Gamma_0(36))$, where $K = \mathbb{Q}(\sqrt{-3})$, $f_c = (2\sqrt{-3})$. This can be found in [1].

(2) $b = 2$; $f_b(4z) = \eta(4z)^2 \eta(8z)^2 = \varphi_{K,c}(z) \in S_2(\Gamma_0(32))$, where $K = \mathbb{Q}(\sqrt{-1})$, $f_c = (2(1+i))$.

(3) $b = 3$; $f_b(3z) = \eta(3z)^2 \eta(9z)^2 = \varphi_{K,c}(z) \in S_2(\Gamma_0(27))$, where $K = \mathbb{Q}(\sqrt{-3})$, $f_c = (3)$.

(4) $b = 4$; $f_b(12z) = \eta(12z)^2 \eta(48z)^2 = \frac{1}{8}(\varphi_{K,c_-}(z) - \varphi_{K,c_+}(z)) \in S_2(\Gamma_0(576))$, where $K = \mathbb{Q}(\sqrt{-1})$, $f_c = (12)$.

(5) $b = 16$; $f_b(12z) = \eta(12z)^2 \eta(16 \cdot 12z)^2 = \frac{1}{16}(\varphi_{603} - \varphi_{203}) - \frac{\sqrt{1-2\sqrt{-6}}}{16 \cdot (6-2\sqrt{-6})}(\varphi_{130} - \varphi'_{130} + \varphi_{310} - \varphi'_{310}) \in S_2(\Gamma_0(144 \cdot 16))$, where φ_{603} and φ_{203} are associated to $K = \mathbb{Q}(\sqrt{-1})$, $f_c = (24)$, and φ_{130} , φ'_{130} , φ_{310} , φ'_{310} are associated to $K = \mathbb{Q}(\sqrt{-6})$, $f_c = (4\sqrt{-6})$.

Proof First we need to compute the specific Hecke character on each case. On case (2), the hecke character c is defined as follows: choose $a + bi$ to be primary, that is, if $b \equiv 0 \pmod{4}$, then $a \equiv 1 \pmod{4}$; if $b \equiv 2 \pmod{4}$, then $a \equiv 3 \pmod{4}$. Then $c((a+bi)) = a+bi$. On case (3), the hecke character c is defined as follows: choose $a+b\omega$ to be primary, that is, $a \equiv 2 \pmod{3}$, $b \equiv 0 \pmod{3}$. Then $c((a+b\omega)) = -(a+b\omega)$. On case (4), the hecke character c_+ and c_- is defined as follows: choose $a + bi$ to be primary, $a + bi \equiv (1-i)^u \pmod{3}$, $u = 0, 1, \dots, 7$, and $a + bi \equiv (-1+2i)^v \pmod{4}$, $v = 0, 1$. Then $c_+((a+bi)) = i^u \cdot (-1)^v \cdot (a+bi)$, $c_-((a+bi)) = (-i)^u \cdot (-1)^v \cdot (a+bi)$. Case (5) is a little bit complicated. To φ_{603} and φ_{203} , the hecke character c_{603} and c_{203} are defined as: choose $a + bi$ to be "standard", which means $a + bi \equiv 1, 2 + i, 3 + 4i, 2 + 3i, 4 + i, -1 - 2i, 3i$, or $-3 - 2i \pmod{8}$, if $a + bi \equiv (1-i)^u \pmod{3}$, $u = 0, 1, \dots, 7$, and $a + bi \equiv (2+i)^v(4+i)^w \pmod{8}$, $v = 0, 1, 2, 3$, $w = 0, 1, 2, 3$. Then $c_{rst}(a+bi) = \zeta_8^{ru} \cdot i^{sv} \cdot i^{tw} \cdot (a+bi)$, $\zeta_8 = e^{\frac{2\pi i}{8}}$. To φ_{130} , φ'_{130} , φ_{310} , φ'_{310} , the hecke character c_{130} , c'_{130} , c_{310} and c'_{310} are defined as: choose $a + b\sqrt{-6}$ to be "standard",

which means $a+b\sqrt{-6} \equiv 1, 1+\sqrt{-6}, 1-\sqrt{-6}, 5$, or the product of them (mod $4\sqrt{-6}$), if $a+b\sqrt{-6} \equiv (1+\sqrt{-6})^u \cdot (1-\sqrt{-6})^v \cdot 5^w$, $u, v = 0, 1, 2, 3$, $w = 0, 1$. Then $c_{rst}(a+b\sqrt{-6}) = i^{ru} \cdot i^{sv} \cdot (-1)^{tw} \cdot (a+b\sqrt{-6})$. However, $\mathbb{Q}(\sqrt{-6})$ is not a Principal Ideal Domain, that means not every ideal in $\mathbb{Q}(\sqrt{-6})$ is principal. Let $5 = (5, 2+\sqrt{-6}) \cdot (5, 2-\sqrt{-6})$, and denote $(5, 2+\sqrt{-6})$ as α . Every non-principal ideal plus α should be a principal ideal because $\mathbb{Q}(\sqrt{-6})$ has Class Number 2. Let $c(\alpha) = \sqrt{c(\alpha^2)} = \sqrt{c(-1+2\sqrt{-6})}$, and $c'(\alpha) = -\sqrt{c(-1+2\sqrt{-6})}$. That would make the definition explicit.

Next we will verify the above result by Sturm's Theorem[7]:

Sturm's Theorem Suppose that N is a positive integer, p is prime, and $f(z), g(z) \in M_k(\Gamma_0(N), \chi) \cap \mathbb{Z}[[q]]$. If

$$\text{ord}_p(f(z) - g(z)) > \frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma_0(N)]$$

in which

$$[SL_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{\ell \text{ prime: } \ell|N} \left(1 + \frac{1}{\ell}\right)$$

then $f(z) \equiv g(z) \pmod{p}$.

Case (1)(2)(3) is trivial, because they are in one-dimension linear spaces. To case (4) and (5), although it is very complicated to get the above result, it is relatively easy to check it. By Sturm's Theorem, we only need to verify the first 192 and 768 coefficients, which are listed in appendix 2 and appendix 3. \square

Here we make an example to show how to compute the coefficients of the case $b = 16$. Let $n = 29645 = 5 * 7^2 * 11^2$. Denote $\varphi_{uvw} = \sum a_{uvw}(n)q^n$, and $\varphi'_{uvw} = \sum a'_{uvw}(n)q^n$. $5 = (2+i)(-1-2i)$ in $\mathbb{Q}(\sqrt{-1})$, so $a_{603}(5) = 2$, $a_{203}(5) = -2$. By the formula $a(p^r) = a(p)a(p^{r-1}) - p \cdot a(p^{r-2})$, we can get $a_{603}(49) = a_{203}(49) = -7$, $a_{603}(121) = a_{203}(121) = -11$, so $a_{603}(n) = 2 * 7 * 11$, $a_{203}(n) = -2 * 7 * 11$. Next, $5 = (5, 2+\sqrt{-6}) \cdot (5, 2-\sqrt{-6})$ in $\mathbb{Q}(\sqrt{-6})$, so $a_{130}(5) = a_{310}(5) = \frac{6-2\sqrt{-6}}{\sqrt{1-2\sqrt{-6}}}$, $a'_{130}(5) = a'_{310}(5) = -\frac{6-2\sqrt{-6}}{\sqrt{1-2\sqrt{-6}}}$. There are three ideals whose norm are 49: $(-5+2\sqrt{-6})$, $(-5-2\sqrt{-6})$ and (7) , so $a_{130}(49) = a'_{130}(49) = a_{310}(49) = a'_{310}(49) = 17$. There are also three ideals whose norm are 121: $(5+4\sqrt{-6})$, $(5-4\sqrt{-6})$ and (11) , so $a_{130}(121) = a'_{130}(121) = a_{310}(121) = a'_{310}(121) = 21$. So $a_{130}(29645) = a_{310}(29645) = 17 * 21 * \frac{6-2\sqrt{-6}}{\sqrt{1-2\sqrt{-6}}}$, $a'_{130}(29645) = a'_{310}(29645) = -17 * 21 * \frac{6-2\sqrt{-6}}{\sqrt{1-2\sqrt{-6}}}$. Put all these data in the formula, we get $a(29645) = \frac{1}{4}(7 * 11 - 17 * 21) = -70$. Verifying it by directly computing with computer, we can see this is the right result.

Acknowledgement

The author is grateful to my tutor Chunlei Liu for the guide throughout the article and the help from Haobo Dai.

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Appendix 1

This is the first 1000 coefficients of $\prod_{n=1}^{\infty} (1 - q^n)^2 = \sum_{n=1}^{\infty} b(n)q^n$. The first row, from left to right, are $b(1), b(2), \dots, b(23)$, the second row are $b(24), b(25), \dots, b(46)$, and so on. We can see that in the first six rows, except for the 21st column, there are at least two numbers not equal to zero in each column. That's just what we need in Lemma 3.3.

-2	-1	2	1	2	-2	0	-2	-2	1	0	0	2	3	-2	2	0	0	-2	-2	0	0	-2
-1	0	2	2	-2	2	1	2	0	2	-2	-2	2	0	-2	0	-4	0	0	0	1	-2	0
0	2	0	2	2	1	-2	0	2	2	0	0	-2	0	-2	0	-2	2	0	-4	0	0	-2
-1	2	0	2	0	0	0	-2	2	4	1	0	0	2	-2	2	-2	0	0	2	0	-2	0
-2	-2	0	-2	0	0	0	2	-2	-1	-2	-2	4	0	0	2	0	2	0	0	0	3	-2
0	0	4	2	0	-2	0	0	-2	0	0	-2	0	2	0	-2	0	-2	-2	-2	0	0	2
-2	-1	2	0	0	0	2	-2	0	-2	2	2	2	2	0	1	2	-2	0	0	0	0	2
0	0	0	0	2	0	-2	-2	-4	2	0	0	-2	0	-2	0	2	0	-2	0	0	-4	1
-2	0	0	4	2	-2	2	0	0	0	-2	4	0	-2	2	1	0	2	2	0	0	2	0
0	-4	2	0	0	-2	0	0	2	0	-2	0	0	0	0	-2	-2	-4	-2	2	0	2	0
0	0	-2	-1	0	2	0	-2	0	0	0	0	2	0	0	2	0	4	2	-2	0	-1	2
-2	2	0	0	0	2	-2	4	0	0	2	-2	0	0	-2	-2	0	-2	0	0	0	2	-2
0	0	0	-2	-2	0	0	0	0	-2	0	-2	-2	1	0	0	2	2	2	0	0	-2	-4
4	2	0	-2	0	0	2	0	2	2	3	-2	0	2	2	0	-2	0	0	0	0	-2	0
2	2	0	0	-2	0	0	0	0	0	0	0	-2	-4	0	2	-4	0	-2	0	0	2	0
2	0	-2	0	-2	0	-3	0	0	2	-2	0	2	0	0	0	0	2	0	4	0	0	0
0	2	-4	0	2	1	0	2	2	-4	2	2	0	0	2	0	2	0	0	-2	0	0	-2
0	0	-2	0	-2	0	0	2	2	2	-2	0	-4	-2	0	0	0	-2	0	-2	0	-2	2
0	-2	0	0	0	1	0	0	2	0	-2	2	0	0	0	-4	0	0	2	2	0	2	0
0	0	2	0	0	4	3	-2	0	0	0	0	0	-2	0	-2	2	0	4	0	0	0	2
0	0	-2	-2	2	0	0	0	-4	-2	2	0	0	-2	-2	0	2	-2	-2	0	0	0	0
4	-2	0	0	-2	-2	-2	0	0	-4	1	4	-2	0	0	0	0	0	2	2	0	0	2
0	0	0	-2	2	0	0	0	0	0	0	-2	2	1	6	0	2	0	-2	0	0	2	0
-2	2	0	2	0	-2	0	0	0	-2	-2	0	0	0	2	0	2	0	-2	0	0	0	0
2	-4	-2	-2	0	-4	2	0	-2	0	0	0	-2	2	0	0	2	-2	0	0	-2	1	0
0	2	0	-2	-2	0	0	-2	0	0	4	0	2	-2	4	0	0	0	-2	4	0	0	2
0	0	2	1	-2	0	0	0	0	-2	2	2	-2	0	0	2	2	2	0	2	0	0	0
0	-2	-2	-4	0	0	2	2	-2	-2	0	0	0	-2	0	0	0	-2	2	2	0	-2	0
-2	-2	0	0	2	0	-4	-2	0	0	0	-2	0	0	0	-1	2	0	4	-4	0	2	0
2	0	0	-2	0	0	2	2	0	-2	0	0	-2	0	0	0	2	2	2	0	0	2	3
2	0	2	-2	0	-2	0	-2	2	0	0	2	0	0	2	-4	0	0	0	2	0	0	0
0	2	-4	0	0	2	2	0	0	-2	-2	0	-2	-2	2	-4	-2	0	0	0	0	2	-2
0	0	-2	0	2	0	-4	2	2	0	0	0	0	-2	-2	-1	0	-2	0	0	0	0	2
0	-2	0	2	0	2	0	0	4	0	2	0	0	-2	0	0	0	-2	2	2	0	0	0
2	3	-2	-2	0	2	0	0	0	0	0	4	0	0	0	4	0	2	-2	0	0	-2	-2
0	0	-2	0	-2	2	0	0	-2	2	0	0	-2	-2	4	0	0	0	0	-2	0	-2	0
2	0	-2	0	-4	-2	0	0	0	0	-2	0	-2	0	0	0	2	0	0	2	0	-1	0
2	0	-4	0	-2	0	0	6	2	-2	0	-2	-2	0	0	0	2	0	2	0	-2	0	0

[illegible]

Appendix 2

Let $f_4(12z) = \sum a(n)q^n$, and $\varphi_{K,c_+}(z) = \sum b(n)q^n$, $\varphi_{K,c_-}(z) = \sum c(n)q^n$. By proposition 4.1(4), we only need to verify $a(n) = \frac{1}{8}(c(n) - b(n))$.

We omit those n that $2|n$ or $3|n$, because those coefficients are all zero.

n	$a(n)$	$b(n)$	$c(n)$
1	0	1	1
5	1	-4	4
7	0	0	0
11	0	0	0
13	0	6	6
17	-2	8	-8
19	0	0	0
23	0	0	0
25	0	11	11
29	-1	4	-4
31	0	0	0
35	0	0	0
37	0	2	2
41	2	-8	8
43	0	0	0
47	0	0	0
49	0	-7	-7
53	-1	4	-4
55	0	0	0
59	0	0	0
61	0	10	10
65	6	-24	24
67	0	0	0
71	0	0	0
73	0	6	6
77	0	0	0
79	0	0	0
83	0	0	0
85	0	-32	-32
89	-4	16	-16
91	0	0	0
95	0	0	0
97	0	-18	-18
101	-5	20	-20
103	0	0	0
107	0	0	0
109	0	-18	-18
113	-4	16	-16

n	$a(n)$	$b(n)$	$c(n)$
115	0	0	0
119	0	0	0
121	0	-11	-11
125	6	-24	24
127	0	0	0
131	0	0	0
133	0	0	0
137	-2	8	-8
139	0	0	0
143	0	0	0
145	0	-16	-16
149	5	-20	20
151	0	0	0
155	0	0	0
157	0	-22	-22
161	0	0	0
163	0	0	0
167	0	0	0
169	0	23	23
173	1	-4	4
175	0	0	0
179	0	0	0
181	0	-18	-18
185	2	-8	8
187	0	0	0
191	0	0	0

Appendix 3

Let $f_{16}(12z) = \sum a(n)q^n$, and $\varphi_{603}(z) = \sum b_1(n)q^n$, $\varphi_{203}(z) = \sum b_2(n)q^n$, $\frac{\sqrt{1-2\sqrt{-6}}}{6-2\sqrt{-6}}\varphi_{130}(z) = \sum c_1(n)q^n$, $\frac{\sqrt{1-2\sqrt{-6}}}{6-2\sqrt{-6}}\varphi'_{130}(z) = \sum c_2(n)q^n$, $\frac{\sqrt{1-2\sqrt{-6}}}{6-2\sqrt{-6}}\varphi_{310}(z) = \sum c_3(n)q^n$, $\frac{\sqrt{1-2\sqrt{-6}}}{6-2\sqrt{-6}}\varphi'_{310}(z) = \sum c_4(n)q^n$. Denote $\frac{\sqrt{1-2\sqrt{-6}}}{6-2\sqrt{-6}} = t$. By proposition 4.1(5), we need to verify that $a(n) = \frac{1}{16}(b_1(n) - b_2(n)) - \frac{1}{16}(c_1(n) - c_2(n) + c_3(n) - c_4(n))$.

We omit those n that $2|n$ or $3|n$, because those coefficients are all zero.

"*" means that because of the "minus" in the formula, $* - * = 0$, so we don't need to compute these specific values. These case are:

- (1) If $p = (a + bi)$ is an ideal in $\mathbb{Q}(\sqrt{-1})$, $3|a$ or $3|b$, then $c_+(p) = c_-(p)$.
- (2) If p is a principle ideal in $\mathbb{Q}(\sqrt{-6})$, then $c_{130}(p) = c'_{130}(p)$, $c_{310}(p) = c'_{310}(p)$.

n	$a(n)$	$b_1(n)$	$b_2(n)$	$c_1(n)$	$c_2(n)$	$c_3(n)$	$c_4(n)$
1	0	1	1	1	1	1	1
5	0	2	-2	1	-1	1	-1
7	0	0	0	$-2\sqrt{6}t$	$-2\sqrt{6}t$	$2\sqrt{6}t$	$2\sqrt{6}t$
11	0	0	0	$\frac{4}{\sqrt{6}}$	$-\frac{4}{\sqrt{6}}$	$-\frac{4}{\sqrt{6}}$	$\frac{4}{\sqrt{6}}$
13	0	4	4	0	0	0	0
17	1	8	-8	0	0	0	0
19	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0
25	0	-1	-1	$7t$	$7t$	$7t$	$7t$
29	-2	-10	10	3	-3	3	-3
31	0	0	0	$-2\sqrt{6}t$	$-2\sqrt{6}t$	$2\sqrt{6}t$	$2\sqrt{6}t$
35	0	0	0	$-2\sqrt{6}$	$2\sqrt{6}$	$2\sqrt{6}$	$-2\sqrt{6}$
37	0	12	12	0	0	0	0
41	-1	-8	8	0	0	0	0
43	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0
49	0	-7	-7	$17t$	$17t$	$17t$	$17t$
53	2	14	-14	-1	1	-1	1
55	0	0	0	*	*	*	*
59	0	0	0	$\frac{8}{\sqrt{6}}$	$-\frac{8}{\sqrt{6}}$	$-\frac{8}{\sqrt{6}}$	$\frac{8}{\sqrt{6}}$
61	0	12	12	0	0	0	0
65	1	8	-8	0	0	0	0
67	0	0	0	0	0	0	0
71	0	0	0	0	0	0	0
73	0	6	6	$14t$	$14t$	$14t$	$14t$
77	2	0	0	-8	8	-8	8
79	0	0	0	$6\sqrt{6}t$	$6\sqrt{6}t$	$-6\sqrt{6}t$	$-6\sqrt{6}t$
83	0	0	0	$-\frac{4}{\sqrt{6}}$	$\frac{4}{\sqrt{6}}$	$\frac{4}{\sqrt{6}}$	$-\frac{4}{\sqrt{6}}$
85	0	16	16	0	0	0	0

n	$a(n)$	$b_1(n)$	$b_2(n)$	$c_1(n)$	$c_2(n)$	$c_3(n)$	$c_4(n)$
89	-2	-16	16	0	0	0	0
91	0	0	0	0	0	0	0
95	0	0	0	0	0	0	0
97	0	18	18	$2t$	$2t$	$2t$	$2t$
101	0	-2	2	-1	1	-1	1
103	0	0	0	$-6\sqrt{6}t$	$-6\sqrt{6}t$	$6\sqrt{6}t$	$6\sqrt{6}t$
107	0	0	0	$\frac{8}{\sqrt{6}}$	$-\frac{8}{\sqrt{6}}$	$-\frac{8}{\sqrt{6}}$	$\frac{8}{\sqrt{6}}$
109	0	20	20	0	0	0	0
113	-2	-16	16	0	0	0	0
115	0	0	0	0	0	0	0
119	0	0	0	0	0	0	0
121	0	-11	-11	$21t$	$21t$	$21t$	$21t$
125	-2	-12	12	2	-2	2	-2
127	0	0	0	$2\sqrt{6}t$	$2\sqrt{6}t$	$-2\sqrt{6}t$	$-2\sqrt{6}t$
131	0	0	0	$\frac{16}{\sqrt{6}}$	$-\frac{16}{\sqrt{6}}$	$-\frac{16}{\sqrt{6}}$	$\frac{16}{\sqrt{6}}$
133	0	0	0	0	0	0	0
137	1	8	-8	0	0	0	0
139	0	0	0	0	0	0	0
143	0	0	0	0	0	0	0
145	0	-20	-20	$36t$	$36t$	$36t$	$36t$
149	0	-14	14	-7	7	-7	7
151	0	0	0	$-10\sqrt{6}t$	$-10\sqrt{6}t$	$10\sqrt{6}t$	$10\sqrt{6}t$
155	0	0	0	$-2\sqrt{6}$	$2\sqrt{6}$	$2\sqrt{6}$	$-2\sqrt{6}$
157	0	12	12	0	0	0	0
161	0	0	0	0	0	0	0
163	0	0	0	0	0	0	0
167	0	0	0	0	0	0	0
169	0	3	3	*	*	*	*
173	2	26	-26	5	-5	5	-5
175	0	0	0	*	*	*	*
179	0	0	0	$-\frac{8}{\sqrt{6}}$	$\frac{8}{\sqrt{6}}$	$\frac{8}{\sqrt{6}}$	$-\frac{8}{\sqrt{6}}$
181	0	20	20	0	0	0	0
185	3	24	-24	0	0	0	0
187	0	0	0	0	0	0	0
191	0	0	0	0	0	0	0
193	0	14	14	*	*	*	*
197	-2	-2	2	7	-7	7	-7
199	0	0	0	*	*	*	*
203	0	0	0	$-6\sqrt{6}$	$6\sqrt{6}$	$6\sqrt{6}$	$-6\sqrt{6}$
205	0	-16	-16	0	0	0	0
209	0	0	0	0	0	0	0
211	0	0	0	0	0	0	0
215	0	0	0	0	0	0	0
217	0	0	0	*	*	*	*
221	4	32	-32	0	0	0	0
223	0	0	0	*	*	*	*

n	$a(n)$	$b_1(n)$	$b_2(n)$	$c_1(n)$	$c_2(n)$	$c_3(n)$	$c_4(n)$
227	0	0	0	$\frac{20}{\sqrt{6}}$	$-\frac{20}{\sqrt{6}}$	$-\frac{20}{\sqrt{6}}$	$\frac{20}{\sqrt{6}}$
229	0	4	4	0	0	0	0
233	2	16	-16	0	0	0	0
235	0	0	0	0	0	0	0
239	0	0	0	0	0	0	0
241	0	-30	-30	*	*	*	*
245	-6	-14	14	17	-17	17	-17
247	0	0	0	0	0	0	0
251	0	0	0	$\frac{4}{\sqrt{6}}$	$-\frac{4}{\sqrt{6}}$	$-\frac{4}{\sqrt{6}}$	$\frac{4}{\sqrt{6}}$
253	0	0	0	0	0	0	0
257	-4	-32	32	0	0	0	0
259	0	0	0	0	0	0	0
263	0	0	0	0	0	0	0
265	0	-28	-28	*	*	*	*
269	-4	-26	26	-3	3	-3	3
271	0	0	0	*	*	*	*
275	0	0	0	$\frac{28}{\sqrt{6}}$	$-\frac{28}{\sqrt{6}}$	$-\frac{28}{\sqrt{6}}$	$\frac{28}{\sqrt{6}}$
277	0	*	*	0	0	0	0
281	4	32	-32	0	0	0	0
283	0	0	0	0	0	0	0
287	0	0	0	0	0	0	0
289	0	47	47	-17	-17	-17	-17
293	-2	-34	34	-9	9	-9	9
295	0	0	0	*	*	*	*
299	0	0	0	0	0	0	0
301	0	0	0	0	0	0	0
305	3	24	-24	0	0	0	0
307	0	0	0	0	0	0	0
311	0	0	0	0	0	0	0
313	0	*	*	*	*	*	*
317	4	22	-22	-5	5	-5	5
319	0	0	0	*	*	*	*
323	0	0	0	0	0	0	0
325	0	-4	-4	0	0	0	0
329	0	0	0	0	0	0	0
331	0	0	0	0	0	0	0
335	0	0	0	0	0	0	0
337	0	*	*	0	0	0	0
341	2	0	0	-8	8	-8	8
343	0	0	0	*	*	*	*
347	0	0	0	$-\frac{20}{\sqrt{6}}$	$\frac{20}{\sqrt{6}}$	$\frac{20}{\sqrt{6}}$	$-\frac{20}{\sqrt{6}}$
349	0	*	*	0	0	0	0
353	-2	-16	16	0	0	0	0
355	0	0	0	0	0	0	0
359	0	0	0	0	0	0	0
361	0	-19	-19	*	*	*	*
365	-2	12	-12	14	-14	14	-14

n	$a(n)$	$b_1(n)$	$b_2(n)$	$c_1(n)$	$c_2(n)$	$c_3(n)$	$c_4(n)$
367	0	0	0	*	*	*	*
371	0	0	0	$2\sqrt{6}$	$-2\sqrt{6}$	$-2\sqrt{6}$	$2\sqrt{6}$
373	0	*	*	0	0	0	0
377	-5	-40	40	0	0	0	0
379	0	0	0	0	0	0	0
383	0	0	0	0	0	0	0
385	0	0	0	*	*	*	*
389	6	34	-34	-7	7	-7	7
391	0	0	0	0	0	0	0
395	0	0	0	$6\sqrt{6}$	$-6\sqrt{6}$	$-6\sqrt{6}$	$6\sqrt{6}$
397	0	*	*	0	0	0	0
401	-5	-40	40	0	0	0	0
403	0	0	0	0	0	0	0
407	0	0	0	0	0	0	0
409	0	*	*	*	*	*	*
413	4	0	0	-16	16	-16	16
415	0	0	0	*	*	*	*
419	0	0	0	$-\frac{28}{\sqrt{6}}$	$\frac{28}{\sqrt{6}}$	$\frac{28}{\sqrt{6}}$	$-\frac{28}{\sqrt{6}}$
421	0	*	*	0	0	0	0
425	-1	-8	8	0	0	0	0
427	0	0	0	0	0	0	0
431	0	0	0	0	0	0	0
433	0	*	*	*	*	*	*
437	0	0	0	0	0	0	0
439	0	0	0	*	*	*	*
443	0	0	0	$-\frac{20}{\sqrt{6}}$	$\frac{20}{\sqrt{6}}$	$\frac{20}{\sqrt{6}}$	$-\frac{20}{\sqrt{6}}$
445	0	-32	-32	0	0	0	0
449	5	40	-40	0	0	0	0
451	0	0	0	0	0	0	0
455	0	0	0	0	0	0	0
457	0	*	*	*	*	*	*
461	-2	-38	38	-11	11	-11	11
463	0	0	0	*	*	*	*
467	0	0	0	$-\frac{28}{\sqrt{6}}$	$\frac{28}{\sqrt{6}}$	$\frac{28}{\sqrt{6}}$	$-\frac{28}{\sqrt{6}}$
469	0	0	0	0	0	0	0
473	0	0	0	0	0	0	0
475	0	0	0	0	0	0	0
479	0	0	0	0	0	0	0
481	0	48	48	0	0	0	0
485	4	36	-36	2	-2	2	-2
487	0	0	0	*	*	*	*
491	0	0	0	$-\frac{16}{\sqrt{6}}$	$\frac{16}{\sqrt{6}}$	$\frac{16}{\sqrt{6}}$	$-\frac{16}{\sqrt{6}}$
493	0	-80	-80	0	0	0	0
497	0	0	0	0	0	0	0
499	0	0	0	0	0	0	0
503	0	0	0	0	0	0	0

n	$a(n)$	$b_1(n)$	$b_2(n)$	$c_1(n)$	$c_2(n)$	$c_3(n)$	$c_4(n)$
505	0	-4	-4	*	*	*	*
509	2	-10	10	-13	13	-13	13
511	0	0	0	*	*	*	*
515	0	0	0	$-6\sqrt{6}$	$6\sqrt{6}$	$6\sqrt{6}$	$-6\sqrt{6}$
517	0	0	0	0	0	0	0
521	-5	-40	40	0	0	0	0
523	0	0	0	0	0	0	0
527	0	0	0	0	0	0	0
529	0	-23	-23	*	*	*	*
533	-4	-32	32	0	0	0	0
535	0	0	0	*	*	*	*
539	0	0	0	$\frac{68}{\sqrt{6}}$	$-\frac{68}{\sqrt{6}}$	$-\frac{68}{\sqrt{6}}$	$\frac{68}{\sqrt{6}}$
541	0	*	*	0	0	0	0
545	5	40	-40	0	0	0	0
547	0	0	0	0	0	0	0
551	0	0	0	0	0	0	0
553	0	0	0	*	*	*	*
557	-8	-38	38	13	-13	13	-13
559	0	0	0	0	0	0	0
563	0	0	0	$\frac{20}{\sqrt{6}}$	$-\frac{20}{\sqrt{6}}$	$-\frac{20}{\sqrt{6}}$	$\frac{20}{\sqrt{6}}$
565	0	-32	-32	0	0	0	0
569	-5	-40	40	0	0	0	0
571	0	0	0	0	0	0	0
575	0	0	0	0	0	0	0
577	0	*	*	*	*	*	*
581	-2	0	0	8	-8	8	-8
583	0	0	0	*	*	*	*
587	0	0	0	$\frac{32}{\sqrt{6}}$	$-\frac{32}{\sqrt{6}}$	$-\frac{32}{\sqrt{6}}$	$\frac{32}{\sqrt{6}}$
589	0	0	0	0	0	0	0
593	2	16	-16	0	0	0	0
595	0	0	0	0	0	0	0
599	0	0	0	0	0	0	0
601	0	*	*	*	*	*	*
605	-8	-22	22	21	-21	21	-21
607	0	0	0	*	*	*	*
611	0	0	0	0	0	0	0
613	0	*	*	0	0	0	0
617	4	32	-32	0	0	0	0
619	0	0	0	0	0	0	0
623	0	0	0	0	0	0	0
625	0	-19	-19	*	*	*	*
629	12	96	-96	0	0	0	0
631	0	0	0	*	*	*	*
635	0	0	0	$2\sqrt{6}$	$-2\sqrt{6}$	$-2\sqrt{6}$	$2\sqrt{6}$
641	1	8	-8	0	0	0	0
643	0	0	0	0	0	0	0
647	0	0	0	0	0	0	0

n	$a(n)$	$b_1(n)$	$b_2(n)$	$c_1(n)$	$c_2(n)$	$c_3(n)$	$c_4(n)$
649	0	0	0	*	*	*	*
653	2	26	-26	5	-5	5	-5
655	0	0	0	*	*	*	*
659	0	0	0	$-\frac{32}{\sqrt{6}}$	$\frac{32}{\sqrt{6}}$	$\frac{32}{\sqrt{6}}$	$-\frac{32}{\sqrt{6}}$
661	0	*	*	0	0	0	0
665	0	0	0	0	0	0	0
667	0	0	0	0	0	0	0
671	0	0	0	0	0	0	0
673	0	*	*	*	*	*	*
677	4	2	-2	-15	15	-15	15
679	0	0	0	*	*	*	*
637	0	-28	-28	0	0	0	0
683	0	0	0	$-15i$	$15i$	$15i$	$-15i$
685	0	16	16	0	0	0	0
689	7	56	-56	0	0	0	0
691	0	0	0	0	0	0	0
695	0	0	0	0	0	0	0
697	0	-64	-64	0	0	0	0
701	-4	-10	10	11	-11	11	-11
703	0	0	0	0	0	0	0
707	0	0	0	$2\sqrt{6}$	$-2\sqrt{6}$	$-2\sqrt{6}$	$2\sqrt{6}$
709	0	*	*	0	0	0	0
713	0	0	0	0	0	0	0
715	0	0	0	0	0	0	0
719	0	0	0	0	0	0	0
721	0	0	0	*	*	*	*
725	-4	10	-10	21	-21	21	-21
727	0	0	0	*	*	*	*
731	0	0	0	0	0	0	0
733	0	*	*	0	0	0	0
737	0	0	0	0	0	0	0
739	0	0	0	0	0	0	0
743	0	0	0	0	0	0	0
745	0	-28	-28	*	*	*	*
749	4	0	0	*	*	*	*
751	0	0	0	*	*	*	*
755	0	0	0	$-10\sqrt{6}$	$10\sqrt{6}$	$10\sqrt{6}$	$-10\sqrt{6}$
757	0	*	*	0	0	0	0
761	5	40	-40	0	0	0	0
763	0	0	0	0	0	0	0
767	0	0	0	0	0	0	0

DEPARTMENT OF MATHEMATICS, SHANGHAI JIAO TONG UNIVERSITY, SHANGHAI 200240,
E-mail address: wyd007001@sjtu.edu.cn